VIBRATIONS OF THIN PLANE WINGS IN TANDEM IN PLANE INCOMPRESSIBLE FLOW

(KOLEBANIIA TONKOGO POLIPLANA TANDEM V PLOSKOM NESZHIMAEMOM POTOKE)

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This paper deals with small vibrations of thin plane wings arranged in tandem in plane incompressible flow. The solution is obtained by breaking the problem down into two simpler ones [1]. One of these problems is nonhomogeneous and represents noncirculatory flow past a system of wings, whilst the second is a homogeneous problem which can be solved by means of functional combinations containing several constants. Thin wing theory is used to solve these problems [2], the whole investigation consisting essentially of finding the constants by means of linear equations.

Closer attention is given to the problem of vibrations in a tandem biplane system in which one of the wings is fixed. Approximate expressions are given for the hydrodynamic forces and the energy characteristics of the system, regarded as a moving group.

1. Kinematic relations for vibrating planes in tandem. We consider a system of thin plane wings of infinite span in tandem which is undergoing small harmonic vibrations of frequency σ in a stream of incompressible fluid moving with constant velocity v_0. We introduce the coordinate system Oxy (Fig.1) which moves with the undisturbed stream at velocity v_0; then if we assume the disturbed fluid flow to be irrotational, the velocity potential $\Phi(x, y, t)$ of the total motion satisfies the linearized flow conditions

$$\frac{\partial \Phi}{\partial y} = v_{nk}(x)\, e^{j\sigma t} \qquad \text{at } a_k b_k \qquad (k = 1, \ldots, n;\ j = \sqrt{-1}) \tag{1.1}$$

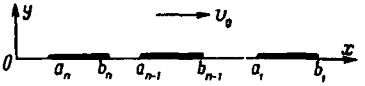

Fig. 1.

Both here, and in what follows, we shall only deal with the real part (with respect to the imaginary number j) of complex expressions which involve the exponential time term. Moreover, for brevity, but without restricting the validity of expression (1.1) it is assumed that the average angle of incidence is zero. If the thin wings vibrate as solid bodies the complex amplitude of the normal velocity $v_{nk}(x)$ can be determined from the expression

$$v_{nk}(x) = v_k + \left(\frac{j}{\mu_0} + x\right)\omega_k \qquad \left(\mu_0 = \frac{\sigma}{v_0}\right) \tag{1.2}$$

where v_k and ω_k are the complex amplitudes of the vertical and angular velocities of the plate $a_k b_k$.

Assuming the disturbed vibrational motion of the fluid to be quasi-stationary we find

$$\Phi(x, y, t) = \varphi(x, y)e^{j\sigma t}, \qquad \frac{\partial\varphi}{\partial y} = v_{nk}(x) \qquad \text{on } a_k b_k \tag{1.3}$$

The harmonic function $\psi(x, y)$ must satisfy several additional conditions. In fact, for the fluid pressure we have the linearized expression (p_0 and ρ are respectively the pressure and density in the undisturbed fluid)

$$p - p_0 = \rho v_0\left(\frac{\partial\varphi}{\partial x} - j\mu_0\varphi\right)e^{j\sigma t} \tag{1.4}$$

Thus from symmetry considerations and the condition of continuity of pressure we obtain

$$\begin{gathered}\varphi(x, -y) = -\varphi(x, y), \qquad \varphi(x, 0) = 0 \quad \text{for } x > b_1 \\ \frac{\partial\varphi}{\partial x} - j\mu_0\varphi = 0 \quad \text{for } y = 0 \text{ outside } a_k b_k \ (x < b_1)\end{gathered} \tag{1.5}$$

It follows from these conditions that, behind the wing-trailing edges, lines of discontinuity in horizontal velocity exist, representing a vortex sheet originating at these trailing edges. The solution we are trying to find should satisfy the condition of finite velocity at the trailing edges a_k.

To solve the problem we introduce the function $w = \phi' + i\psi$ of the complex variable $z = x + iy$, where the imaginary number $i = \sqrt{-1}$ is not interchangeable with the imaginary number j. Later on we shall split the above problem into the two simpler problems; $w(z) = w_0(z) + w_1(z)$ where the functions $w_0 = \phi_0 + i\psi_0$ and $w_1 = \phi_1 + i\psi_1$, satisfy conditions

$$\begin{gathered}\operatorname{Im}\frac{dw_0}{dz} = -v_{nk}(x) \quad \text{on } a_k b_k, \qquad \operatorname{Re} w_0(x) = 0 \text{ outside } a_k b_k \\ \operatorname{Im} w_1(x) = A_k \quad \text{on } a_k b_k, \qquad \operatorname{Re} w_1(x) = 0 \quad \text{for } x > b_1\end{gathered} \tag{1.6}$$

$$\operatorname{Re}\left(\frac{dw_1}{dz} - j\mu_0 w_1\right) = 0 \qquad \text{for } y = 0 \text{ outside } a_k b_k \quad (x < b_1) \tag{1.7}$$

The function $w_0(z)$ represents the complex potential of fluid flow without circulation, while the function $w_1(z)$ is a solution to the homogeneous problem and is linear in terms of the constants A_k. It is obvious that the function $w(z)$, determined from conditions (1.6) and (1.7), satisfies (1.3) and (1.5). The constants A_k in this function must be found from the condition of finite velocity at the trailing edges a_k.

Using thin wing theory [2.3] we find immediately that

$$\frac{dw_0}{dz} = \frac{1}{2\pi i g(z)}\left(\sum_{k=0}^{n-2} B_k z + 2\sum_{k=1}^{n}\int_{a_k}^{b_k}\frac{v_{nk}(\xi)\, g_k(\xi)}{\xi - z}\, d\xi\right) \tag{1.8}$$

$$g(z) = \left(\prod_{s=1}^{n}(z - a_s)(z - b_s)\right)^{1/2}$$

$$g_k(x) = \left((b_k - x)(x - a_k)\prod_{s\neq k}^{n}(x - a_s)(x - b_s)\right)^{1/2} \tag{1.9}$$

The constants $B_k (k = 0, 1, \ldots, n-2)$, which are real with respect to i, are found by equating to zero the circulation round $n-1$ sections $a_k b_k$

$$\int_{a}^{b_l}\sum_{k=0}^{n-2} B_k x^k \frac{dx}{g(x)} + 2\int_{a}^{b_l}\frac{dx}{g_l(x)}\sum_{k=1}^{n}\int_{a_k}^{b_k}\frac{v_{nk}(\xi)\, g_k(\xi)}{\xi - x}\, d\xi = 0 \qquad (l = 1, \ldots, n-1) \tag{1.10}$$

To determine w_1 we introduce a further function

$$f(z) = r + is = \frac{dw_1}{dz} - j\mu_0 w_1 \tag{1.11}$$

From (1.7) we have the following conditions for the function $f(z)$:

$$\operatorname{Im} f(x) = -j\mu_0 A_k \quad \text{on } a_k b_k, \qquad \operatorname{Re} f(x) = 0 \text{ outside } a_k b_k \tag{1.12}$$

It follows from these conditions that the expansion of the function $f(z)$ in the neighborhood of a point at infinity takes the following form (α_k is real with respect to i)

$$f(z) = \frac{i\alpha_1}{z} + \frac{i\alpha_2}{z^2} + \ldots \tag{1.13}$$

The function $f(z)$ is thus determined, and so is dw_0/dz. In fact, using the expansion (1.13), we shall have (C_k is real with respect to i)

$$f(z) = \frac{1}{2\pi i g(z)}\left(\sum_{k=0}^{n-1} C_k z^k + 2j\mu_0\sum_{k=0}^{n} A_k\int_{a_k}^{b_k}\frac{g_k(\xi)}{\xi - z}\, d\xi\right) \tag{1.14}$$

To determine the constants C_k and A_k we make use of the condition of finite fluid velocity at the trailing edges a_k, which can be written

$$\lim_{z\to a_l}(z-a_l)^{1/2}\left(\frac{dw_0}{dz}+f\right)=0 \qquad (l=1,\ldots,n)$$

Applying this condition we arrive at the system of linear equations

$$\sum_{k=0}^{n-1} C_k a_l^k + 2j\mu_0 \sum_{k=1}^{n} A_k \int_{a_k}^{b_k} \frac{g_k(\xi)}{\xi-a_l}\,d\xi +$$

$$+\sum_{k=0}^{n-2} B_k a_l^k + 2\sum_{k=1}^{n}\int_{a_k}^{b_k}\frac{v_{nk}(\xi)\,g_k(\xi)}{\xi-a_l}\,d\xi = 0 \qquad (l=1,\ldots,n) \tag{1.15}$$

in which all the constants C_k are given linearly in terms of A_k. To determine the latter we must satisfy the first of the conditions (1.7). To do this let us regard (1.11) as a differential equation in w_k. Bearing in mind that at large distances in front of the planes the fluid is undisturbed we find

$$\psi_1(x,0)=e^{j\mu_0 x}\int_{\infty}^{x} e^{-j\mu_0 x}\, s(x,0)\,dx \tag{1.16}$$

Making use of this expression and satisfying the first of the conditions (1.7) we have the following system of linear equations

$$A_l = e^{j\mu_0 b_l}\Big[\int_{\infty}^{b_l} e^{-j\mu_0 x}\, s(x,0)\,dx + \sum_{s=1}^{l-1} A_s\left(e^{-j\mu_0 a_s}-e^{-j\mu_0 b_s}\right) +$$

$$+\sum_{s=1}^{l-1}\int_{a_s}^{b_{s+1}} e^{-j\mu_0 x}\, s(x,0)\,dx\Big] \qquad (l=1,\ldots,n) \tag{1.17}$$

In this way, the final determination of the function $w(z)$ consists of finding the constants A_k, B_k and C_k by means of the linear equations (1.10), (1.15) and (1.17). However, an analysis of transient motion of multiplanes in tandem [3,4] in the general case, with arbitrary variation of normal velocities with time, leads us into great difficulty because some unwieldy integral equations have to be solved.

2. Vibrations of one wing in a tandem biplane. We now discuss a tandem biplane, the edges of which are determined thus: $-a_1 = -a$, $b_1 = a$, $a_2 = -c-b$ and $b_2 = -c+b$, where c is the abscissa of the center of gravity of the second wing, and $2b$ is the width of this wing. We shall now assume the second wing to be at rest ($v_{n2}(x)=0$)*, then formulas (1.8), (1.9), and (1.14) for this case take the form

* In all our calculations we assume $c>0$, i.e. the stationary wing is at the rear. If the stationary wing is in front ($c<0$) the following change should be made in the formulas, $(c^2-a^2)^{1/2} = -(|c^2-a^2|)^{1/2}$

$$\frac{dw_0}{dz} = \frac{1}{2\pi i g(z)}\left(B_0 + 2\int_{-a}^{a} \frac{v_n(\xi) g_1(\xi)}{\xi - z}\, d\xi\right) \tag{2.1}$$

$$f(z) = \frac{1}{2\pi i g(z)}\left(C_0 + C_1 z + 2j\mu_0 A_1 \int_{-a}^{a} \frac{g_1(\xi)}{\xi - z}\, d\xi + 2j\mu_0 A_2 \int_{-c-b}^{-c+b} \frac{g_2(\xi)}{\xi - z}\, d\xi\right) \tag{2.2}$$

$$g(z) = ((z^2 - a^2)[(z + c)^2 - b^2])^{1/2}, \quad g_1(x) = ((a^2 - x^2)\ [(x + c)^2 - b^2])^{1/2}$$
$$g_2(x) = -(|x^2 - a^2|\ [b^2 - (x + c)^2])^{1/2} \tag{2.3}$$

To determine the constants we have equations (1.10), (1.15) and (1.17), which in particular give

$$B_0 \int_{-a}^{a} \frac{dx}{g_1(x)} + 2\int_{-a}^{a} \frac{dx}{g_1(x)} \int_{-a}^{a} \frac{v_n(\xi) g_1(\xi)}{\xi - x}\, d\xi = 0 \tag{2.4}$$

$$B_0 + C_0 - C_1 a + 2\int_{-a}^{a} \frac{v_n(\xi) + j\mu_0 A_1}{\xi + a} g_1(\xi)\, d\xi + 2j\mu_0 A_2 \int_{-c-b}^{-c+b} \frac{g_2(\xi)}{\xi + a}\, d\xi = 0 \tag{2.5}$$

$$B_0 + C_0 - C_1(c + b) + 2\int_{-a}^{a} (v_n(\xi) + j\mu_0 A_1)\left((a^2 - \xi^2)\frac{\xi + c - b}{\xi + c + b}\right)^{1/2} d\xi -$$
$$- 2j\mu_0 A_2 \int_{-c-b}^{-c+b} \left(|\xi^2 - a^2|\frac{b - (\xi + c)}{b + \xi + c}\right)^{1/2} d\xi = 0 \tag{2.6}$$

$$A_1 = e^{j\mu_0 a}\int_{\infty}^{a} e^{-j\mu_0 x} s(x, 0)\, dx \tag{2.7}$$

$$A_2 = A_1 e^{j\mu_0 (a+b-c)} + e^{j\mu_0 (b-c)} \int_{-a}^{-c+b} e^{-j\mu_0 x} s(x, 0)\, dx \tag{2.8}$$

Let us suppose that the stationary wing is much shorter than the vibrating one so that $c \gg b$. In this case equations (2.4) - (2.8) allow us to find explicit approximate expressions for the constants entering these equations, if we represent the stationary wing as a concentrated singularity. In fact, it follows from (2.1) and (2.4) that outside the section $(-c - b,\ -c + b)$ the following expressions are valid to accuracy of the order of $(b/c)^2$.

$$B_0 = -2\int_{-a}^{a} v_n(\xi)(a^2 - \xi^2)^{1/2}\, d\xi, \quad \frac{dw_0}{dz} = \frac{1}{\pi i (z^2 - a^2)^{1/2}} \int_{-a}^{a} \frac{v_n(\xi)(a^2 - \xi^2)^{1/2}}{\xi - z}\, d\xi \tag{2.9}$$

Expressions (2.9) represent fluid flow without circulation round an isolated thin vibrating wing. It therefore follows that the effect of the stationary wing on the noncirculatory part of the fluid flow close to the vibrating wing, which gives rise to the supplementary effect of connected masses, is only manifest in dipole, quadripole and higher approximations.

Equations (2.5) and (2.6) appear in the following simplified form with this degree of accuracy

$$C_0 - C_1 a + 2\pi j \mu_0 a^2 \left(\frac{c}{a} - \frac{1}{2}\right) A_1 = (c - a) D_1 \tag{2.10}$$

$$C_0 - C_1 c + \pi j \mu_0 a^2 A_1 = b\,(D_2 + C_1 + 2\pi j \mu_0 c A_1 + \\ + 4 j \mu_0 (3 - 2^{1/2}) (c^2 - a^2)^{1/2} A_2) \tag{2.11}$$

$$D_1 = -2 \int_{-a}^{a} v_n(\xi) \left(\frac{a - \xi}{a + \xi}\right)^{1/2} d\xi, \quad D_2 = 2 \int_{-a}^{a} \frac{v_n(\xi)\,(a^2 - \xi^2)^{1/2}}{\xi + c}\, d\xi \tag{2.12}$$

If follows from (2.2) that outside the section $(-c-b,\ -c+b)$, to an accuracy of terms of order $(b/c)^2$, the following expression prevails

$$f(z) = \frac{1}{2\pi i\,(z^2 - a^2)^{1/2}} \left[C_1 - 2\pi j \mu_0 A_1 (z - (z^2 - a^2)^{1/2})\right] + \frac{C_0 - C_1 c + \pi j \mu a^2 A_1}{2\pi i\,(z + c)\,(z^2 - a^2)^{1/2}} \tag{2.13}$$

while it follows from (2.11), that the second factor in (2.13) is of order b/c. Thus the approximate expressions (2.9) and (2.13) correspond to representation of the effect of the stationary wing on the fluid flow close to the vibrating wing, by that of a concentrated vortex of given strength.

Using expressions (2.7) and (2.13) we can obtain the following equations to determine the constant A_1:

$$C_1 H_0^{(2)}(\mu) - 2\pi\mu A_1 H_1^{(2)}(\mu) = \frac{2j}{\pi a} E_0 (C_0 - C_1 c + \pi j \mu a A_1) \qquad (\mu = \mu_0 a) \tag{2.14}$$

$$E_0 = \int_{\infty}^{1} \frac{e^{-j\mu x}\, dx}{(x + \alpha_0)(x^2 - 1)^{1/2}}, \qquad \frac{dE_0}{d\mu} - j\alpha_0 E_0 = \frac{\pi}{2} H_0^{(2)}(\mu) \qquad \left(\alpha_0 = \frac{c}{a}\right) \tag{2.15}$$

Here $H_n^{(2)} = I_n - jN_n$ is a Hankel function, and from relation (2.15) we obtain

$$E_0 = \frac{\ln(\alpha_0 - (\alpha_0^2 - 1)^{1/2})}{(\alpha_0^2 - 1)^{1/2}} e^{j\nu} + \frac{\pi}{2\alpha_0} e^{j\nu} \left[H_c^{(2)}\left(\frac{1}{\alpha_0},\ \nu\right) - j H_s^{(2)}\left(\frac{1}{\alpha_0},\ \nu\right)\right] \quad (\nu = \mu_0 c) \tag{2.16}$$

where the functions $H_c^{(2)}$ and $H_s^{(2)}$ have been tabulated [5] and are expressed in terms of Hankel functions by means of relations

$$H_c^{(2)}\left(\frac{1}{\alpha_0},\ \nu\right)=\int_0^\nu H_0^{(2)}\left(\frac{x}{\alpha_0}\right)\cos x\,dx,\qquad H_s^{(2)}\left(\frac{1}{\alpha_0},\ \nu\right)=\int_0^\nu H_0^{(2)}\left(\frac{x}{\alpha_0}\right)\sin x\,dx \tag{2.17}$$

Equations (2.10), (2.11) and (2.14) allow us to determine the constants C_i and A_i to an accuracy of order $(b/c)^{(2)}$. To do this, we put

$$C_i=C_{i0}+bC_{i1},\qquad A_i=A_{i0}+bA_{i1} \tag{2.18}$$

Then from these equations, we obtain

$$C_{i0}=C(\mu)D_1,\qquad 2\pi\mu A_{10}=(1-C(\mu))D_1$$
$$C_{00}=a\left[(\alpha_0 C(\mu))-\tfrac{1}{2}(1-C(\mu))\right]D_1$$

$$C_{01}=\Big[(2\alpha_0-1)C(\mu)+\left(\alpha_0-\tfrac{3}{2}\right)(1-C(\mu))-$$
$$-\frac{2}{\pi}\left(\alpha_0+\tfrac{1}{2}\right)(\alpha_0-1)E_0T(\mu)\Big]\frac{G}{\alpha_0-1} \tag{2.19}$$
$$C_{11}=\left[C(\mu)-\frac{2}{\pi}(\alpha_0-1)\,E_0T(\mu)\right]\frac{G}{a(\alpha_0-1)}$$
$$2\pi j\mu A_{11}=\left[1-C(\mu)+\frac{2}{\pi}(\alpha_0-1)E_0T(\mu)\right]\frac{G}{a(\alpha_0-1)}$$

Here $C(\mu)$, $T(\mu)$ and G represent the following expressions

$$C(\mu)=\frac{H_1^{(2)}(\mu)}{H_1^{(2)}(\mu)+jH_0^{(2)}(\mu)},\qquad T(\mu)=\frac{1}{H_1^{(2)}(\mu)+jH_0^{(2)}(\mu)} \tag{2.20}$$
$$G=D_2+2\pi j\nu A_{10}+C_{10}+4j\mu A_{20}(3-2^{1/2})(\alpha_0^2-1)^{1/2}$$

For the final determination of all the constants we must find the value of A_{20}. From equation (2.8) and expressions (2.13) and (2.19) we find

$$A_{20}=\left[\frac{1}{2\pi j\mu}(1-C(\mu))(1-e^{-j\nu}E_2)-\frac{1}{2\pi}e^{-j\nu}E_1C(\mu)\right]D_1 \tag{2.21}$$

$$E_1=\int_1^{\alpha_0}\frac{e^{-j\mu x}}{(x^2-1)^{1/2}}\,dx,\qquad E_2=\frac{dE_1}{d\mu} \tag{2.22}$$

Note that the functions E_1 and E_2 can be evaluated in terms of Bessel functions $I_k(\mu)$. Actually, if we make the substitution $U=x-(x^2-1)^{1/2}$ the expression for E_1 takes the following form

$$E_1=-\int_1^{U_0}\exp\left[\frac{j\mu}{2}\left(U+\frac{1}{U}\right)\right]\frac{dU}{U}\qquad (U_0=\alpha_0-(\alpha_0^2-1)^{1/2})$$

Now, making use of the expansion

$$\exp\left[\frac{j\mu}{2}\left(U+\frac{1}{U}\right)\right]=I_0(\mu)+\sum_{k=1}^{\infty}j^k(U^k+U^{-k})I_k(\mu)$$

we arrive at the following expression for E_1, which is convenient for calculation,

$$E_1 = -I_0(\mu)\ln U_0 - \sum_{k=1}^{\infty}\frac{j^k}{k} I_k(\mu)\,(U_0^{\ k} - U_0^{\ -k}) \qquad (2.23)$$

3. Hydrodynamic forces which act on the tandem biplane. We now work out the hydrodynamic forces which act on the vibrating wing in the tandem biplane. If we use expression (1.4) and represent the function w in terms of w_0 and f, we arrive at the following relations for lift and moment:

$$Y = \rho v_0 e^{j\sigma t}\int_K \left(f(z) + j\mu_0 z\frac{dw_0}{dz}\right)dz$$

$$M = \rho v_0 e^{j\sigma t}\int_K \left[z f(z) + z\left(1 + \frac{j\mu_0 z}{2}\right)\frac{dw_0}{dz}\right]dz \qquad (3.1)$$

where K is the contour which bounds the section $(-a, a)$ and is anticlockwise.

From formulas (2.9), (2.13) and (2.19) and the theorem of residues we find

$$Y = Y_0 + \frac{b}{a}Y_1, \qquad M = M_0 + \frac{b}{a}M_1 \qquad (3.2)$$

where Y_0 and M_0 are the lift and moment respectively corresponding to the vibration of an isolated wing:

$$Y_0 = -2\rho v_0\int_{-a}^{a}\left[C(\mu)\left(\frac{a-x}{a+x}\right)^{1/2} + j\mu\left(1 - \frac{x^2}{a^2}\right)^{1/2}\right]v_n(x, t)\,dx$$

$$(v_n(x, t) = v_n(x)\,e^{j\sigma t}) \qquad (3.3)$$

$$M_0 = -2\rho a v_0\int_{-a}^{a}\left[\left(1 - \frac{x^2}{a^2}\right)^{1/2} + \frac{j\mu x}{2a}\left(1 - \frac{x^2}{a^2}\right)^{1/2} + \frac{1}{2}(C(\mu) - 1)\left(\frac{a-x}{a+x}\right)^{1/2}\right]v_n(x, t)\,dx$$

and $Y_1 b/a$ and $M_1 b/a$ are the supplementary force and moment caused by the action of the stationary wing as a concentrated vortex.

$$Y_1 = \rho v_0 G e^{j\sigma t}\left(\frac{C(\mu)}{\alpha_0 - 1} - \frac{2}{\pi}E_0 T(\mu) + (\alpha_0^2 - 1)^{-1/2}\right)$$

$$M_1 = \rho v_0 a G e^{j\sigma t}\left(\frac{\alpha_0 - 3/2}{\alpha_0 - 1} + \frac{C(\mu)}{2(\alpha_0 - 1)} - \frac{1}{\pi}E_0 T(\mu) - \alpha_0(\alpha_0^2 - 1)^{-1/2}\right) \qquad (3.4)$$

To calculate the suction forces at the leading edges b_k, we use the general formula [2]

$$X_k = -\rho\pi \lim_{z \to b_k} \left\{ (z - b_k) \left[\frac{dw}{dz} e^{j\sigma t} \right]^2 \right\} \qquad (3.5)$$

Using (2.9), (2.13) and (2.19), we get an expression for the mean value of the suction at the leading edge $z = a$ of the vibrating wing

$$X_1^* = X_{10}^* + \frac{b}{a} X_{11}^* \qquad \left(A^* = \frac{\sigma}{2\pi} \int_t^{t+2\pi/\sigma} A(t)\, dt \right) \qquad (3.6)$$

where X_{10}^* is the average suction on an isolated vibrating wing

$$X_{10}^* = \frac{\rho a}{\pi} \left| -\int_{-a}^{a} \frac{v_n(x)\, dx}{(a^2 - x^2)^{1/2}} + \frac{1}{2a} (C(\mu) - 1) D_1 \right|^2 \qquad (3.7)$$

and $X_{11}^* b/a$ is the supplementary component in the suction expression caused by the stationary wing acting as a concentrated vortex.

$$X_{11}^* = \frac{\rho}{2\pi} \operatorname{Re} \left[G \left(\frac{2C(\mu) - 1}{\alpha_0 - 1} - \frac{4}{\pi} E_0 T(\mu) + \frac{1}{\alpha_0 + 1} \right) \overline{F} \right]$$

$$F = -\int_{-a}^{a} \frac{v_n(x)\, dx}{(a^2 - x^2)^{1/2}} + \frac{1}{2a} (C(\mu) - 1) D_1 \qquad (3.8)$$

To work out the suction forces acting on the stationary wing, we must start with the accurate formulas (2.1), (2.2) and (2.5) and after this we must transform the boundary. The result of this gives

$$X_2^* = \frac{\rho b}{4\pi (c^2 - a^2)} |G|^2 \qquad (3.9)$$

The complete expression for the projection of the hydrodynamic forces on the x axis takes the form ($\beta(t)$ is the angle of incidence of the vibrating wing):

$$T = X_1 + X_2 - Y\beta \qquad \left(\beta(t) = -\frac{j}{\sigma} \omega^{j\sigma t} \right) \qquad (3.10)$$

For several modes of vibration $T^* > 0$ and in this case a tensile force appears, i.e. the vibrating wing in the biplane tandem system can be regarded as a moving group. In this case the average value of useful and wasted power is given by the

$$E^* = T^* v_0, \quad N^* = -(YV + M\Omega)^* \qquad (V = ve^{j\sigma t},\ \Omega = \omega e^{j\sigma t}) \qquad (3.11)$$

The relations obtained above allow one to work out these energy characteristics and the thrust efficiency $\eta = E^*/N^*$.

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